ANSWERS AND EXPLANATIONS TO SECTION I

Problem 1. If $g(x)=\frac{1}{32}x^4-5x^2$, find $g'(4)$.

First, take the derivative.

$$g'(x)=\frac{1}{32}\left(4x^3\right)-5(2x)=\frac{x^3}{8}-10x$$

Now, Plug In 4 for x.

$$\frac{(4)^3}{8}-10(4)=8-40=-32$$

The answer is (B).

Problem 2. The domain of the function $f(x)=\sqrt{4-x^2}$ is

When you have a square root in a function, the domain will require that the expression under the radical (the "radicand") not be negative. Thus, the domain will be those values where $4-x^2$ is not negative.

In other words, $4-x^2\geq 0$.

We solve this by, first, factoring the expression on the left. $(2+x)(2-x)\geq 0$

Next, we take the roots of the left side, which are –2 and 2, and put them on a number line.

Now, we pick a value in each of the three regions on the number line $x<-2$, $-2<x<2$, and $x>2$. We plug the value into the expression $4-x^2$ to see if we get a positive or negative value. If it's positive, then we include that region in the domain. If it's negative, then we exclude that region from the domain.

Let's try -3 for a value in the region $x<-2$.

We get: $4-(-3)^2=-5$, so we exclude the region $x<-2$ from the domain.

Now, we try 0 for a value in the region $-2<x<2$.

We get: $4-(0)^2=4$, so we include the region $-2<x<2$ in the domain.

Finally, we try 3 for a value in the region $x>2$.

We get: $4-(3)^2=-5$, so we exclude the region $x>2$ from the domain.

Because the radicand is allowed to be zero, we include the endpoints in the domain. Therefore, the domain is $-2\leq x\leq 2$.

The answer is (D).

Problem 3. $\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$ is

Notice that if we plug 5 into the expressions in the numerator and the denominator, we get: $\frac{0}{0}$, which is undefined. Before we give up, we need to see if we can simplify the limit so that it can be evaluated. If we factor the expression in the numerator, we get: $\frac{(x+5)(x-5)}{(x-5)}$, which can be simplified to $x + 5$.

Now, if we take the limit (by Plugging In 5 for x), we get 10.

The answer is (B).

Problem 4. If $f(x) = \frac{x^5 - x + 2}{x^3 + 7}$, find $f'(x)$.

We need to use the Quotient Rule, which is:

Given $f(x) = \frac{g(x)}{h(x)}$ then $f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$

Here, we have:

$$f'(x) = \frac{(x^3 + 7)(5x^4 - 1) - (x^5 - x + 2)(3x^2)}{(x^3 + 7)^2}$$

The answer is (E).

Problem 5. Evaluate $\lim_{h \to 0} \frac{5\left(\frac{1}{2} + h\right)^4 - 5\left(\frac{1}{2}\right)^4}{h}$.

Notice how this limit takes the form of the definition of the Derivative, which is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Here, if we think of $f(x)$ as $5x^4$, then this expression gives the derivative of $5x^4$ at the point $x = \frac{1}{2}$.

The derivative of $5x^4$ is $f'(x) = 20x^3$.

At $x = \frac{1}{2}$, we get $f'\left(\frac{1}{2}\right) = 20\left(\frac{1}{2}\right)^3 = \frac{5}{2}$

The answer is (A).

Problem 6. $\int x\sqrt{3x}\,dx =$

First, rewrite the integral as: $\int x \cdot \sqrt{3} \cdot x^{\frac{1}{2}}\,dx$.

Now we can simplify the integral to: $\sqrt{3}\int x^{\frac{3}{2}}\,dx$.

Using the power rule for integrals, which is $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$,

we get: $\sqrt{3}\int x^{\frac{3}{2}}\,dx = \sqrt{3}\frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C = \frac{2\sqrt{3}}{5}x^{\frac{5}{2}} + C$

The answer is (A).

Problem 7. Find k so that $f(x) = \begin{cases} \dfrac{x^2-16}{x-4} & ; x \neq 4 \\ k & ; x = 4 \end{cases}$ is continuous for all x.

In order for $f(x)$ to be continuous at a point c, there are three conditions that need to be fulfilled:

(1) $f(c)$ exists.

(2) $\lim_{x \to c} f(x)$ exists.

(3) $\lim_{x \to c} f(x) = f(c)$.

First, let's check condition (1). $f(4)$ exists; it's equal to k.

Next, let's check condition (2). From the left side, we get:

$$\lim_{x \to 4^-} \frac{x^2-16}{x-4} = \lim_{x \to 4^-} \frac{(x-4)(x+4)}{x-4} = \lim_{x \to 4^-}(x+4) = 8$$

From the right side, we get:

$$\lim_{x \to 4^+} \frac{x^2-16}{x-4} = \lim_{x \to 4^+} \frac{(x-4)(x+4)}{x-4} = \lim_{x \to 4^+}(x+4) = 8$$

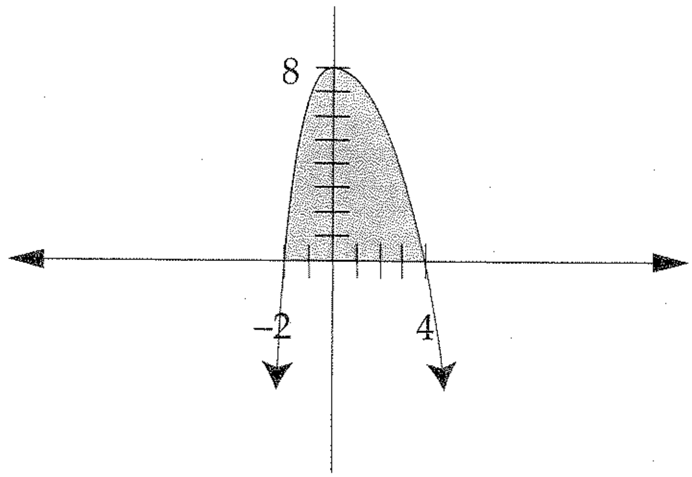
Therefore, the limit exists and $\lim_{x \to 4} \frac{x^2-16}{x-4} = 8$.

Now, let's check condition (3). In order for this condition to be fulfilled, k must equal 8.

The answer is (D).

Problem 8. Which of the following integrals correctly gives the area of the region consisting of all points above the x-axis and below the curve $y = 8 + 2x - x^2$?

The curve $y = 8 + 2x - x^2$ is an upside-down parabola and looks like this:



Notice that it crosses the x-axis at $x = -2$ and at $x = 4$.

The formula for the area of the region under the curve $f(x)$ and above the x-axis from $x = a$ to $x = b$ is: $\int_a^b f(x)\,dx$.

Thus, in order to find the area of the desired region, we need to evaluate the integral $\int_{-2}^{4} \left(8 + 2x - x^2\right) dx$.

The answer is (C).

Problem 9. If $f(x) = x^2 \cos 2x$, find $f'(x)$.

Here we need to use the product rule, which is:

If $f(x) = uv$, where u and v are both functions of x,

$$\text{then } f'(x) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Here, we get:

$$f'(x) = x^2(-2\sin 2x) + 2x(\cos 2x)$$

The answer is (D).

PROBLEM 10. An equation of the line tangent to $y = 4x^3 - 7x^2$ at $x = 3$ is

If we want to find the equation of the tangent line, first we need to find the y-coordinate that corresponds to $x = 3$. It is:

$$y = 4(3)^3 - 7(3)^2 = 108 - 63 = 45.$$

Next, we need to find the derivative of the curve at $x = 3$. It is

$$\frac{dy}{dx} = 12x^2 - 14x \text{ and at } x = 3,\ \left.\frac{dy}{dx}\right|_{x=3} = 12(3)^2 - 14(3) = 66$$

Now we have the slope of the tangent line and a point that it goes through. We can use the point-slope formula for the equation of a line, $(y - y_1) = m(x - x_1)$, and Plug In what we have just found. We get:

$$(y - 45) = 66(x - 3)$$

The answer is (B).

PROBLEM 11. $\int_0^{\frac{1}{2}} \frac{2}{\sqrt{1-x^2}}\,dx =$

This integral is of the form $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$, where $a = 1$.

Thus, we get:

$$\int_0^{\frac{1}{2}} \frac{2dx}{\sqrt{1-x^2}} = 2\sin^{-1}(x)\Big|_0^{\frac{1}{2}} = 2\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right] = 2\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{3}$$

The answer is (B).

PROBLEM 12. Find a positive value c that satisfies the conclusion of the Mean Value Theorem for Derivatives for $f(x) = 3x^2 - 5x + 1$ on the interval $[2,5]$.

The Mean Value Theorem for Derivatives says that, given a function $f(x)$ which is continuous and differentiable on $[a,b]$, then there exists some value c on (a,b) where

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Here, we have $\frac{f(b) - f(a)}{b - a} = \frac{f(5) - f(2)}{5 - 2} = \frac{51 - 3}{3} = 16$

and $f'(c) = 6c - 5$, so we simply set $6c - 5 = 16$

If we solve for c, we get: $c = \frac{7}{2}$

The answer is (E).

PROBLEM 13. Given $f(x)=2x^2-7x-10$ find the absolute maximum of $f(x)$ on $[-1,3]$.

First, let's take the derivative and then set it equal to zero to determine any critical points of the function.

$$f'(x)=4x-7$$

$$4x-7=0$$

$$x=\frac{7}{4}$$

Now we can use the second derivative test to determine if this is a local minimum or maximum.

$$f''(x)=4$$

Because the second derivative is always positive, the function is concave up everywhere and thus $x=\frac{7}{4}$ must be a local minimum.

How, then, do we find the absolute maximum? Anytime we are given a function that is defined on an interval, the endpoints of the interval are also critical points. Thus, all that we have to do now is to plug the endpoints into the function and see which one gives us the bigger value. That will be the absolute maximum.

$$f(-1)=2(-1)^2-7(-1)-10=-1$$

$$f(3)=2(3)^2-7(3)-10=-13$$

Therefore, the absolute maximum of $f(x)$ on the interval $[-1,3]$ is -1.

The answer is (A).

PROBLEM 14. Find $\frac{dy}{dx}$ if $x^3y+xy^3=-10$.

We need to use implicit differentiation to find $\frac{dy}{dx}$.

$$3x^2y+x^3\frac{dy}{dx}+y^3+3xy^2\frac{dy}{dx}=0$$

Now, in order to isolate $\frac{dy}{dx}$, we move all of the terms that do not contain $\frac{dy}{dx}$ to the right side of the equals sign:

$$x^3\frac{dy}{dx}+3xy^2\frac{dy}{dx}=-3x^2y-y^3$$

Factor out $\frac{dy}{dx}$:

$$\frac{dy}{dx}\left(x^3+3xy^2\right)=-3x^2y-y^3$$

And divide both sides by $\left(x^3+3xy^2\right)$ to isolate $\frac{dy}{dx}$:

$$\frac{dy}{dx}=-\frac{3x^2y+y^3}{x^3+3xy^2}$$

The answer is (D).

PROBLEM 15. If $f(x)=\sqrt{1+\sqrt{x}}$, find $f'(x)$.

First rewrite the equation using fractional powers instead of radical signs.

$$f(x)=\left(1+x^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

Now take the derivative:

$$f'(x)=\frac{1}{2}\left(1+x^{\frac{1}{2}}\right)^{-\frac{1}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

This can be rewritten as:

$$f'(x)=\frac{1}{4}\frac{1}{\sqrt{x}}\frac{1}{\sqrt{1+\sqrt{x}}}$$

The answer is (D).

PROBLEM 16. $\int 7xe^{3x^2}dx=$

We can use u-substitution to evaluate the integral.

Let $u=3x^2$ and $du=6xdx$. If we solve the second term for $x\,dx$, we get:

$$\frac{1}{6}du=xdx$$

Now we can rewrite the integral as:

$$\frac{7}{6}\int e^u\,du$$

Evaluate the integral to get:

$$\frac{7}{6}e^{u}+C$$

Now substitute back to get:

$$\frac{7}{6}e^{3x^2}+C$$

The answer is (C).

Problem 17. Find the equation of the tangent line to $9x^2+16y^2=52$ through $(2,-1)$.

First, we need to find $\frac{dy}{dx}$. It's simplest to find it implicitly:

$$18x+32y\frac{dy}{dx}=0$$

Now solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx}=-\frac{18x}{32y}=-\frac{9x}{16y}$$

Next, Plug In $x=2$ and $y=-1$ to get the slope of the tangent line at the point:

$$\frac{dy}{dx}=\frac{-18}{-16}=\frac{9}{8}$$

Now use the point-slope formula to find the equation of the tangent line:

$$(y+1)=\frac{9}{8}(x-2)$$

If we multiply through by 8, we get: $8y+8=9x-18$ or $9x-8y-26=0$.

The answer is (B).

Problem 18. A particle's position is given by $s=t^3-6t^2+9t$. What is its acceleration at time $t=4$?

Acceleration is the second derivative of position with restrict to time (Velocity is the first derivative).

The first derivative is: $v(t)=3t^2-12t+9$

The second derivative is: $a(t)=6t-12$

Now we simply Plug In $t=4$ and we get: $a(4)=24-12=12$

The answer is (E).

Problem 19. If $f(x) = 3^{\pi x}$, then $f'(x) =$

The derivative of an expression of the form a^u, where u is a function of x, is:

$$\frac{d}{dx}a^u = a^u \cdot \ln a \cdot \frac{du}{dx}$$

Here, we get:

$$\frac{d}{dx}3^{\pi x} = 3^{\pi x} \cdot \ln 3 \cdot \pi$$

The answer is (E).

Problem 20. The average value of $f(x) = \frac{1}{x}$ from $x = 1$ to $x = e$ is

In order to find the average value, we use the Mean Value Theorem for Integrals, which says that the average value of $f(x)$ on the interval $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

Here, we have $\frac{1}{e-1}\int_1^e \frac{1}{x}\,dx$.

Evaluating the integral, we get: $\ln x\Big|_1^e = \ln e - \ln 1 = 1$. Therefore, the answer is $\frac{1}{e-1}$.

The answer is (E).

Problem 21. If $f(x) = \sin^2 x$, find $f'''(x)$.

We just use the chain rule three times.

$$f'(x) = 2\sin x \cos x = \sin 2x$$

$$f''(x) = 2\cos 2x$$

$$f'''(x) = -4\sin 2x$$

The answer is (D).

Problem 22. Find the slope of the normal line to $y = x + \cos xy$ at $(0,1)$.

First, we need to find $\frac{dy}{dx}$ using implicit differentiation.

$$\frac{dy}{dx} = 1 - \left(x\frac{dy}{dx} + y\right)\sin xy$$

Rather than simplifying this, simply Plug In $(0,1)$ to find $\frac{dy}{dx}$.

We get: $\frac{dy}{dx} = 1$.

This means that the slope of the tangent line at $(0,1)$ is 1, so the slope of the normal line at $(0,1)$ is the negative reciprocal, which is -1.

The answer is (B).

Problem 23. $\int e^x \cdot e^{3x} dx =$

First, add the exponents to get: $\int e^{4x} dx$

Evaluating the integral, we get: $\frac{1}{4}e^{4x} + C$

The answer is (B).

Problem 24. $\lim_{x \to 0} \frac{\tan^3(2x)}{x^3} =$

We will need to use the fact that $\lim_{x \to 0} \frac{\sin x}{x} = 1$ to find the limit.

First, rewrite the limit as

$$\lim_{x \to 0} \frac{\sin^3(2x)}{x^3 \cos^3(2x)}$$

Next, break the fraction into:

$$\lim_{x \to 0} \left(\frac{\sin^3(2x)}{x^3} \frac{1}{\cos^3(2x)} \right)$$

Now, if we multiply the top and bottom of the first fraction by 8, we get:

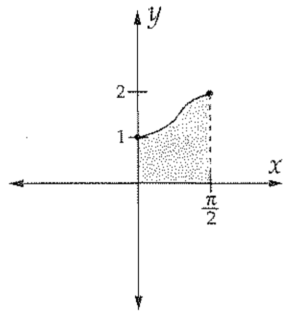
$$\lim_{x \to 0} \frac{8\sin^3(2x)}{(2x)^3} \frac{1}{\cos^3(2x)}$$

Now, we can take the limit, which gives us: $8 \cdot 1 \cdot 1 = 8$.

The answer is (D).

Problem 25. A solid is generated when the region in the first quadrant bounded by the graph of $y = 1+\sin^2 x$, the line $x = \frac{\pi}{2}$, the x-axis, and the y-axis is revolved about the x-axis. Its volume is found by evaluating which of the following integrals?

First, let's graph the curve.



We can find the volume by taking a vertical slice of the region. The formula for the volume of a solid of revolution around the x-axis, using a vertical slice bounded from above by the curve $f(x)$ and from below by $g(x)$, on the interval $[a,b]$, is:

$$\pi\int_a^b \left[f(x)^2 - g(x)^2\right]dx$$

Here, we get:

$$\pi\int_0^{\frac{\pi}{2}} \left(1+\sin^2 x\right)^2 dx$$

The answer is (D).

Problem 26. If $y = \left(\frac{x^3-2}{2x^5-1}\right)^4$, find $\frac{dy}{dx}$ at $x = 1$

We use the Chain Rule and the Quotient Rule.

$$\frac{dy}{dx} = 4\left(\frac{x^3-2}{2x^5-1}\right)^3\left[\frac{\left(2x^5-1\right)\left(3x^2\right)-\left(x^3-2\right)\left(10x^4\right)}{\left(2x^5-1\right)^2}\right]$$

If we Plug In 1 for x, we get:

$$\frac{dy}{dx} = 4(-1)^3\left[\frac{3+10}{1^2}\right] = -52$$

The answer is (A).

PROBLEM 27. $\int x\sqrt{5-x}\,dx =$

We can evaluate this integral using u-substitution.

Let $u = 5 - x$ and $5 - u = x$. Then $-du = dx$.

Substituting, we get:

$$-\int (5-u)u^{\frac{1}{2}}\,du$$

The integral can be rewritten as:

$$-\int \left(5u^{\frac{1}{2}} - u^{\frac{3}{2}}\right) du$$

Evaluating the integral, we get:

$$-5\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + \frac{u^{\frac{5}{2}}}{\frac{5}{2}} + C$$

This can be simplified to:

$$-\frac{10}{3}u^{\frac{3}{2}} + \frac{2}{5}u^{\frac{5}{2}} + C$$

Finally, substituting back, we get:

$$-\frac{10}{3}(5-x)^{\frac{3}{2}} + \frac{2}{5}(5-x)^{\frac{5}{2}} + C$$

The answer is (E).

PROBLEM 28. If $\frac{dy}{dx} = \frac{x^3+1}{y}$ and $y = 2$ when $x = 1$, then, when $x = 2$, $y =$

This is a differential equation that can be solved using separation of variables. Put all of the terms containing y on the left and all of the terms containing x on the right. We get:

$$y\,dy = \left(x^3+1\right)dx$$

Next, we integrate both sides:

$$\int y\,dy = \int \left(x^3+1\right)dx$$

Evaluating the integrals, we get:

$$\frac{y^2}{2} = \frac{x^4}{4} + x + C$$

Next, we Plug In $y = 2$ and $x = 1$ to solve for C. We get: $2 = \frac{1}{4} + 1 + C$ and so $C = \frac{3}{4}$. This gives us:

$$\frac{y^2}{2} = \frac{x^4}{4} + x + \frac{3}{4}$$

Now, if we substitute $x = 2$, we get:

$$\frac{y^2}{2} = 4 + 2 + \frac{3}{4} = \frac{27}{4}$$

Solving for y, we get:

$$y = \pm\sqrt{\frac{27}{2}}$$

The answer is (E).

Problem 29. The graph of $y = 5x^4 - x^5$ has an inflection point (or points) at

In order to find the inflection point(s) of a polynomial, we need to find the values of x where its second derivative is zero.

First, we find the second derivative.

$$\frac{dy}{dx} = 20x^3 - 5x^4$$

$$\frac{d^2y}{dx^2} = 60x^2 - 20x^3$$

Now, let's set the second derivative equal to zero and solve for x.

$$60x^2 - 20x^3 = 0$$

$$20x^2(3 - x) = 0$$

$$x = 3$$

This is the point of inflection. $x = 0$ is not a point of inflection because $\frac{d^2y}{dx^2}$ does not change sign there. If you are unsure that these are correct, graph the function with a calculator and look at the picture.

The answer is (B).