

# 9 Functions and Their Inverses

## Key to Text Coverage

Section	Examples	Exercises	Topics
5.3	1–6	1–107	<i>Inverse functions and their derivatives</i>
5.4	1–10	1–131	<i>Exponential function is inverse of logarithmic function</i>
5.8	1–7	1–82	<i>Inverse trigonometric functions</i>
5.10	5–6	55–62, 64–66	<i>Inverse hyperbolic functions</i>

## Formulas

Let  $y = f(x)$  be a function. Then

- $g = f^{-1}$  is the inverse of  $f$  if  $f(g(x)) = x$  and  $g(f(x)) = x$ .
- The domain of  $f$  is the range of  $f^{-1}$ , and the domain of  $f^{-1}$  is the range of  $f$ .
- The point  $(a, b)$  is on the graph of  $f$  if and only if  $(b, a)$  is on the graph of  $f^{-1}$ .
- $f$  has an inverse if and only if  $f$  is one-to-one.
- If  $g = f^{-1}$  is the inverse of  $f$ , then  $g'(x) = 1/f'(g(x))$ .

## Summary

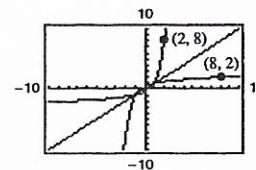
Let  $y = g(x)$  be the inverse of the function  $y = f(x)$ . Then the point  $(a, b)$  is on the graph of  $f$  if and only if the corresponding point  $(b, a)$  is on the graph of  $g$ . Graphically, this means that the graphs of  $f$  and  $g$  are symmetric about the line  $y = x$ . Furthermore, if the slope of the graph of  $f$  at  $(a, b)$  is  $m$ , then the slope at the corresponding point  $(b, a)$  on the graph of  $g$  is  $1/m$ .

It is instructive to verify these facts with a simple example. Let  $f(x) = x^3$  and  $g(x) = x^{1/3}$ . Graph  $f$ ,  $g$  and the line  $y = x$  in the same viewing window. The point  $(2, 8)$  on  $f$  corresponds to  $(8, 2)$  on  $g$ . Notice that the derivatives  $f'(2) = 12$  and  $g'(8) = 1/12$  are reciprocals of each other. What can you say about  $f'(0)$  and  $g'(0)$ ?

You can use implicit differentiation to find the formula for the derivative of an inverse function. For the function  $y = x^3$ , first interchange  $x$  and  $y$ , and then differentiate.

$$\begin{aligned} x &= y^3 \\ 1 &= 3y^2 \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{3y^2} \\ &= \frac{1}{3x^{2/3}} \end{aligned}$$

This is the derivative of  $y = x^{1/3}$ , the inverse of  $y = x^3$ .





### Worked Example

Let  $f(x) = \sqrt{x^2 - 9}$  be defined on the domain  $x < -3$ .

- What is the range of the function  $f$ ?
- Calculate  $f'(x)$  and explain why  $f$  has an inverse.
- Find a formula for the inverse  $g$  of  $f$  and indicate its domain and range.
- Sketch the graph of  $f$  and  $g$  on the same set of axes. Discuss the relationship between the graphs.
- Find the slope of the tangent line to  $f$  at  $(-5, 4)$  and the slope of the tangent line to  $g$  at  $(4, -5)$ .

### SOLUTION

- $f(x) > 0$  for all  $x < -3$ . The range is  $(0, \infty)$ .
- $f'(x) = \frac{1}{2}(x^2 - 9)^{-1/2}(2x) = x/\sqrt{x^2 - 9}$ . Since  $f'(x) < 0$  for all  $x$  in the domain of  $f$ , the graph of  $f$  is decreasing. So,  $f$  is one-to-one and has an inverse.
- You can solve for  $x$  in terms of  $y$ .

$$y = \sqrt{x^2 - 9}$$

$$y^2 = x^2 - 9$$

$$x^2 = y^2 + 9$$

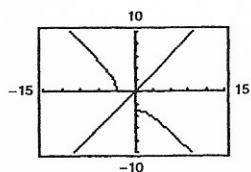
$$x = -\sqrt{y^2 + 9} \quad (\text{select the negative root})$$

$$y = g(x) = -\sqrt{x^2 + 9} \quad (\text{interchange variables})$$

The domain of  $g$  is the range of  $f$ :  $(0, \infty)$ .

The range of  $g$  is the domain of  $f$ :  $(-\infty, -3)$ .

- The graphs of  $f$  and  $g$  are symmetric about the line  $y = x$ , as indicated in the graph.



- The slope of the tangent line to  $f$  at  $(-5, 4)$  is

$$f'(-5) = \frac{-5}{\sqrt{5^2 - 9}} = -\frac{5}{4}$$

The slope of the tangent line to  $g$  at the corresponding point  $(4, -5)$  is therefore the reciprocal,  $-\frac{4}{5}$ .

### Notes

- You can verify this answer by directly calculating the derivative of  $g$ .

$$g'(x) = \frac{1}{2}(x^2 + 9)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + 9}} \Rightarrow g'(4) = \frac{4}{5}$$

Name \_\_\_\_\_

Date \_\_\_\_\_

**Sample Questions**

Show all your work on a separate sheet of paper. Indicate clearly the methods you use because you will be graded on the correctness of your methods as well as on the accuracy of your answers.

**Multiple Choice**

1. Let  $y = mx + b$  be the equation of the tangent line to the graph of  $y = f(x)$  at the point  $(x_0, y_0)$ . What is the slope of the tangent line to the graph of the inverse of  $f$  at the point  $(y_0, x_0)$ ?

(a)  $m$                       (b)  $1/m$                       (c)  $-m$                       (d)  $-1/m$                       (e)  $b$

2. Let  $f(x) = x^3 + 4x - 1$ . If  $g$  is the inverse of  $f$ , find  $g'(4)$ .

(a) 7                      (b)  $1/7$                       (c) 52                      (d)  $1/52$                       (e) 4

3. Which of the following functions defined on  $(-\infty, \infty)$  have an inverse?

I.  $f(x) = 1 - x^5 - 3x^3 - x$

II.  $g(x) = \sin x$

III.  $h(x) = \arctan x$

(a) I only                      (b) II only                      (c) III only                      (d) I and II only                      (e) I and III only

**Free Response**

Let  $f$  be defined by the following integral on the domain  $x > 0$ .

$$f(x) = \int_0^{\ln x} \frac{1}{\sqrt{4 + e^t}} dt$$

- (a) Find  $f(1)$ .  
(b) Find  $f'(1)$ .  
(c) Explain why  $f$  has an inverse.  
(d) Find  $(f^{-1})'(0)$



## SOLUTIONS

## Multiple Choice

- Answer (b). If the slope of the tangent line to the graph of  $y = f(x)$  at  $(x_0, y_0)$  is  $m$ , then the slope at the corresponding point on  $g$  is the reciprocal,  $1/m$ .
- Answer (b). Notice first that  $f(1) = 4$  which means that  $g(4) = 1$ . Now,  $f'(x) = 3x^2 + 4$  and  $f'(1) = 7$ . So,

$$g'(4) = \frac{1}{f'(g(4))} = \frac{1}{f'(1)} = \frac{1}{7}$$

- Answer (e).  $f'(x) = -5x^4 - 9x^2 - 1 < 0$ , which implies that  $f$  is one-to-one and has an inverse. So, (I) is true. The periodic function  $g(x) = \sin x$  does not have an inverse. Recall that the domain of the sine function has to be restricted in order to define the inverse sine function. So, (II) is false. Finally,

$$h'(x) = \frac{1}{1+x^2} > 0$$

which means that the arctangent function has an inverse.

So, (III) is true. In conclusion, only (I) and (III) are true.

## Free Response

$$(a) f(1) = \int_0^{\ln 1} \frac{1}{\sqrt{4+e^t}} dt = \int_0^0 \frac{1}{\sqrt{4+e^t}} dt = 0.$$

- By the Second Fundamental Theorem of Calculus

$$f'(x) = \frac{1}{\sqrt{4+e^{\ln x}}} \frac{1}{x} = \frac{1}{\sqrt{4+x}} \frac{1}{x} \Rightarrow f'(1) = \frac{1}{\sqrt{5}}$$

- The derivative of  $f$  is always positive, which implies that  $f$  is an increasing function. So,  $f$  is one-to-one and has an inverse.
- Because  $f(1) = 0$ , you have  $(f^{-1})(0) = 1$ . So, the derivative is given by the formula

$$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(1)} = \sqrt{5}.$$

It is instructive to verify these results with a graphing utility. You can graph the function  $f$  by graphing  $y = \text{fnInt}(1/\sqrt{4+e^X}, X, 0, \text{LN } X)$  on the viewing window  $[0, 2] \times [-1, 1]$ . Notice that  $y(1) = 0$  confirms the answer to part (a). Furthermore, you can verify that the derivative at  $x = 1$  is  $1/\sqrt{5} \approx 0.44721$ .

