5 The Definite Integral as Total Change

Key to Text Coverage

Section	Examples	Exercises	Topics
4.3	1–5	1–71	*Riemann sums and the definite integral*
4.4	1–8	1–105	*The Fundamental Theorem of Calculus*
5.5	6–7	77–94	*Applications using exponential functions*

Summary

The Fundamental Theorem of Calculus was presented on page 275

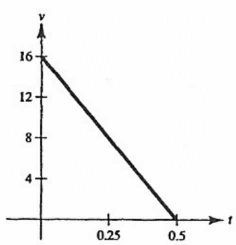
$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

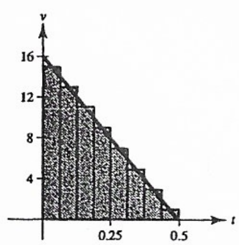
This important theorem states that the definite integral of the rate of change of a quantity, $F'(x)$, gives the total change in that quantity, $F(b) - F(a)$.

One way to illustrate this result is to use an example of velocity. Suppose $v(t) = -32t + 16$ is the velocity of a falling object. The graph of v on the left shows that the velocity of the particle is 16 at time $t = 0$, and 0 at time $t = 1/2$. The total distance traveled by the particle is given by the definite integral

$$\int_a^b v(t)\,dt = \int_0^{1/2} (-32t + 16)\,dt = (-16t^2 + 16t)\Big|_0^{1/2} = -16(1/4) + 8 = 4.$$

In other words, the position function for the particle is $s(t) = -16t^2 + 16t + s_0$ and the total distance traveled is $s(1/2) - s(0) = [-16(1/2)^2 + 16(1/2) + s_0] - [s_0] = 4.$





The definite integral of the velocity is the area under the triangular region. You can think of partitioning this region into a Reimann sum of n rectangles, as shown in the figure at the right. Each rectangle has area (base) × (height) = (time) × (velocity). The sum of the areas of the n rectangles is an approximation of the total distance traveled. Notice that the units for the product (time) × (velocity) are those of distance. This approximation is an example of the Midpoint Rule

$$\text{Area} \approx \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x.$$

For this velocity function

$$\text{Area} \approx \sum_{i=1}^{8} v\left(\frac{t_i + t_{i-1}}{2}\right)\frac{0.5}{8} = \sum_{i=1}^{8}\left[-32\left(\frac{t_i + t_{i-1}}{2}\right) + 16\right]\frac{1}{16}$$

where $t_0 = 0,\ t_1 = \frac{1}{16},\ t_2 = \frac{2}{16}, \ldots,$ and $t_8 = \frac{8}{16}$.

Worked Example

Oil is leaking from a storage tank at the rate $R(t) = 4000e^{-0.3t}$ liters/day.

(a) Graph the curve on the interval $0 \le t \le 20$.

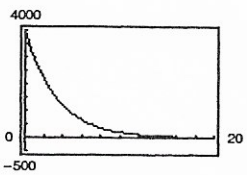
(b) How much oil has leaked on the time interval $[0, 5]$?

(c) How much oil has leaked on the time interval $[5, 10]$? Why is this answer less than that for part (b)?

(d) If the leak is never fixed, approximate the total amount of oil lost.

SOLUTION

(a)



(b) The amount of leaked oil is given by the following definite integral.

$$\int_0^5 4000e^{-0.3t}\,dt = \frac{4000}{-0.3}e^{-0.3t}\Big]_0^5 = \frac{4000}{-0.3}(e^{-0.3(5)} - 1) \approx 10{,}358.26 \text{ liters}$$

(c) Similarly, the amount of leaked oil is given by the integral

$$\int_5^{10} 4000e^{-0.3t}\,dt \approx 2311.24 \text{ liters.}$$

The amount is less than that of part (b) because the rate of leakage diminishes with time.

(d) If the leak is never fixed, then the upper limit of integration approaches infinity. You can approximate the amount of leaked oil by using a large number for this upper limit.

$$\int_0^{1000} 4000e^{-0.3t}\,dt \approx 13{,}333.33 \text{ liters}$$

Notes

(a) $R(t)$ is a **rate of change** and the units are liters per day. If you use the Riemann sum interpretation for finding the area under the graph of this curve, then you can think of summing the areas of little rectangles

$$\int_a^b R(t)\,dt \approx \sum R(t_i)\,\Delta t = \sum(\text{rate})(\text{time}).$$

Notice the units. $R(t)$ is liters per day and Δt is days, so the product is in liters.

(c) At time $t = 0$ the leakage is greatest at 4000 liters per day. This rate diminishes because the exponent of the exponential function is negative.

Name ______________________________

Date ______________________________

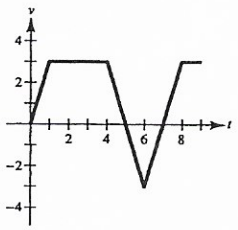
Sample Questions

Show all your work on a separate sheet of paper. Indicate clearly the methods you use because you will be graded on the correctness of your methods as well as on the accuracy of your answers.

Multiple Choice

1. A woman is walking along a straight path beginning at point P. A graph of the walker's velocity is given below in miles per hour. How far away from the starting point P has she walked 9 hours from the starting time?

   (a) 6.5 miles (b) 12 miles (c) 13.5 miles (d) 18 miles (e) 19.5 miles



2. The rate at which water in a pond is evaporating is given by

$$\frac{dV}{dt} = \sqrt{1 + e^{-t}} \text{ liters/day}$$

   where V is the volume in liters and t is the time in days. Use a graphing utility to approximate how much water has evaporated during the first 10 days.

   (a) 1.0 (b) 1.414 (c) 8.531 (d) 9.623 (e) 10.452

3. A car is slowing down according to the following table of velocities.

time (seconds)	0	1	2	3	4
velocity (feet/sec)	25	20	16	12	10

   Use the trapezoidal rule to approximate the distance traveled during these four seconds.

   (a) 65.1 (b) 65.5 (c) 66.5 (d) 67.4 (e) 83.0

4. Use the Midpoint Rule with $n = 2$ to approximate the distance traveled in Question 3.

Free Response

The rate of sales of a product is given by $S(t) = 25e^{kt}$, where t is the time in years with $t = 0$ corresponding to January 1, 1990, and S is measured in thousands of units per year.

(a) What was the rate of sales at the beginning of 1990?

(b) If the rate of sales doubles every 8 years, find the value of k.

(c) Find the average rate of sales over the four year period beginning the first day of 1994.

(d) What is the meaning of the expression $\int_2^4 S(t)\,dt$?

Theme 5

SOLUTIONS

Multiple Choice

1. Answer (c). Notice that the woman reverses direction on the time interval [5, 7]. The distance from the starting point is given by the definite integral

$$\int_0^9 v(t)\, dt$$

which can be calculated from simple geometry. The area above the x-axis is 16.5, from which you subtract the area below the x-axis, $16.5 - 3 = 13.5$.

2. Answer (e). You can use the integration capability of a graphing utility to obtain

$$\int_0^{10} V'(t)\, dt = \int_0^{10} \sqrt{1 + e^{-t}}\, dt \approx 10.452 \text{ liters.}$$

3. Answer (b). Use the trapezoidal rule with $n = 4$.

$$\begin{aligned}
\text{Distance} &= \int_0^4 v(t)\, dt \\
&= \frac{b-a}{2n}[f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)] \\
&= \frac{4}{8}[25 + 2(20) + 2(16) + 2(12) + 10] \\
&= \frac{1}{2}[131] \\
&= 65.5 \text{ feet}
\end{aligned}$$

4. Answer (b).

$$\begin{aligned}
\text{Distance} &= \int_0^4 v(t)\, dt \approx \sum_{i=1}^{2} f\left(\frac{x_i + x_{i-1}}{2}\right) \Delta x \\
&= [f(1)](2) + [f(3)](2) \\
&= (20)2 + (12)2 \\
&= 64
\end{aligned}$$

Free Response

(a) $S(0) = 25e^{k(0)} = 25$ thousand units per year.

(b) $S(8) = 50 = 25e^{k(8)}$ which implies that $8k = \ln 2$ and therefore $k = \frac{1}{8}\ln 2 \approx 0.0866$.

(c) The average value is given by the integral

$$\frac{1}{4}\int_4^8 S(t)\, dt \approx \frac{1}{4}(169.0222) \approx 42.26 \text{ thousand units per year.}$$

(d) This definite integral gives the amount, in thousands of units, of the product that was sold in the two-