

4 The Graphical Relationship Between First and Second Derivatives

Key to Text Coverage

Section	Examples	Exercises	Topics
2.1	1–7	39–45, 49, 50	Relationship between the graphs of f and f'
3.3	1–4	43–48	Increasing and decreasing functions
3.4	1–4	45–56	Concavity and second derivatives
3.6	1–6	1–6, 51–58	General curve sketching principles

Formulas

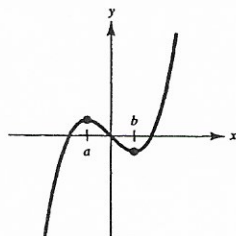
Let $y = f(x)$ be defined on an open interval I containing c .

- f is increasing on I if $x_1 < x_2$ implies $f(x_1) < f(x_2)$.
- f is decreasing on I if $x_1 < x_2$ implies $f(x_1) > f(x_2)$.
- If $f'(x) > 0$ then f is increasing on I .
- If $f'(x) < 0$ then f is decreasing on I .
- If f' changes from negative to positive at c , there is a relative minimum at c .
- If f' changes from positive to negative at c , there is a relative maximum at c .
- f is concave upwards on I if f' is increasing on I .
- f is concave downwards on I if f' is decreasing on I .
- If $f'' > 0$ on I then f is concave upwards on I .
- If $f'' < 0$ on I then f is concave downwards on I .
- The point $(c, f(c))$ is a point of inflection if the concavity changes there.

Summary

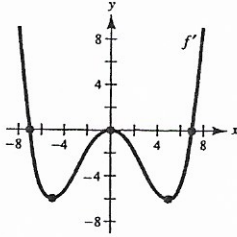
The first and second derivatives of a function provide an enormous amount of useful information about the shape of the graph of the function, as indicated by the properties above. An important skill to develop is that of producing the graph of the derivative of a function, given the graph of the function. Conversely, it is important to be able to produce the graph of a function given the graph of its derivative.

For instance, consider the graph of f below. Because $f'(x) > 0$ on the interval $(-\infty, a)$, f is increasing on that interval. Furthermore $f'(x)$ is decreasing near $x = a$, which implies that the graph of f is concave downwards near $x = a$.



Worked Example

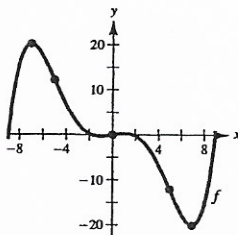
Consider the graph of f' , the derivative of $y = f(x)$ defined on the domain $-9 < x < 9$.



- For what values of x does f have a relative minimum?
- For what values of x does f have a relative maximum?
- Determine the open intervals where the graph of f is concave downwards. Show the analysis that leads to your conclusion.
- Sketch the graph of f on the interval $(-9, 9)$ if $f(0) = 0$. Show the analysis that leads to your graph.

SOLUTION

- There is a relative minimum at $x = 7$.
- There is a relative maximum at $x = -7$.
- The graph of f' is decreasing on the intervals $(-9, -5)$ and $(0, 5)$. By the definition of concavity, this means that the graph of f is concave downwards on these intervals.
- The graph of f is increasing to the left of $x = -7$. The graph is decreasing on the intervals $(-7, 0)$ and $(0, 7)$, and increasing on the interval $(7, 9)$. There is an inflection point at $x = 0$ because the concavity changes from concave upwards to concave downwards.

**Notes**

- The critical numbers are those values of x for which $f'(x) = 0$ or is undefined. From the graph of f' you see that the critical numbers are $x = -7, 0, 7$. Because f' changes from negative to positive at $x = 7$, there is a relative minimum at $x = 7$.
- f' changes from positive to negative at $x = -7$, which means that there is a relative maximum at $x = -7$.
- In a similar manner, f is concave upwards on the intervals $(-5, 0)$, $(5, 9)$. Since the concavity changes at the points $x = -5, 0, 5$, these are the x -coordinates of points of inflection.

Name _____

Date _____



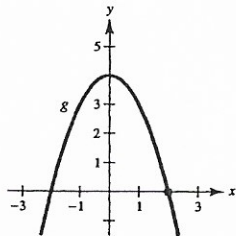
Sample Questions

Show all your work on a separate sheet of paper. Indicate clearly the methods you use because you will be graded on the correctness of your methods as well as on the accuracy of your answers.

Multiple Choice

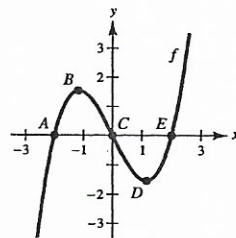
1. Given the graph of $y = g(x)$, estimate the value of $g'(2)$.

- (a) -4 (b) -1 (c) 0 (d) 1 (e) 4



2. At which point A, B, C, D, or E on the graph of $y = f(x)$ are both y' and y'' positive?

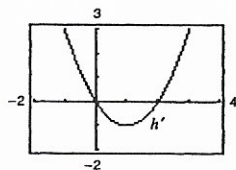
- (a) A (b) B (c) C (d) D (e) E



3. Given the graph of $h'(x)$, which of the following statements are true about the graph of h ?

- I. The graph of h has a point of inflection at $x = 1$.
- II. The graph of h has a relative extremum at $x = 0$.
- III. The graph of h has a relative extremum at $x = 1$.

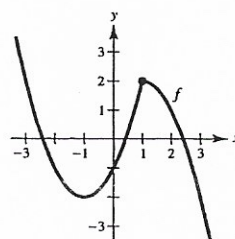
- (a) I only (b) II only (c) III only (d) I and II only (e) I and III only



Free Response

The graph of the function f is shown in the figure.

- (a) Estimate $f'(0)$.
- (b) On what open intervals is f increasing?
- (c) On what open intervals is f concave downwards?
- (d) What are the critical numbers of f ?
- (e) Sketch the graph of f' .



SOLUTIONS

Multiple Choice

- Answer (a). The slope of the tangent line to the graph of $y = g(x)$ is clearly negative. Furthermore, by analyzing the tick marks on the x and y axes you see that the slope is approximately -4 .
- Answer (e). y' is positive at the points where the graph is increasing. y'' is positive at the points where the graph is concave upwards. The only point satisfying these two criteria is point E .
- Answer (d). At $x = 1$ the derivative of h changes from decreasing to increasing. So, the concavity changes at this point, so (I) is true. The derivative changes from positive to negative at $x = 0$ which implies that there is a relative maximum at $x = 0$. So, (II) is true. However, (III) is false because the derivative does not change sign at $x = 1$. In conclusion, only (I) and (II) are true.

Free Response

- $f'(0)$ is approximately 2.
- The graph of f is increasing on the interval $(-1, 1)$.
- The graph of f is concave downwards on the interval $(1, 3)$.
- $x = -1$ is a critical number because $f'(-1) = 0$. Furthermore, $x = 1$ is a critical number because the derivative does not exist there. The derivative from the left is positive, whereas the derivative from the right is 0.
- Because f is not differentiable at $x = 1$, you can expect a break in the graph of f' at this point. The graph of f is decreasing to the left of $x = -1$ and increasing on $(-1, 1)$. The graph of f is decreasing to the right of $x = 1$. So, the graph of f' looks like the following.

