2 Limits of Functions and Unbounded Behavior

Key to Text Coverage

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1.5	1–5	1–75	*Infinite limits and vertical asymptotes*
3.5	1–6	1–88	*Limits at infinity and horizontal asymptotes*

Summary

You can approach limits of functions $\lim_{x\to c} f(x)$ three ways: numerically, graphically, and analytically. For example, to calculate the limit

$$\lim_{x\to 0} \frac{\cos x - 1}{x} \qquad \text{(Section 1.2, Exercise 8)}$$

you can build a table of values for x near 0, graph the function

$$f(x) = \frac{\cos x - 1}{x}$$

and evaluate the limit analytically by multiplying the numerator and denominator by $(\cos x + 1)$. An example of this technique is given by the solution of Exercise 120, Section 1.3.

Functions can exhibit two kinds of unbounded behavior: infinite limits (vertical asymptotes, Section 1.5) and limits at infinity (horizontal asymptotes, Section 3.5). Again, graphical, numerical, and analytical techniques should all be used.

A common error in finding vertical asymptotes is to assume that they occur at points where the denominator vanishes. For instance, the function

$$f(x) = \frac{x^2 + 2x}{x^2 - 4}$$

has a vertical asymptote $x = 2$, but not at $x = -2$. The graph of f is the same as that of $g(x) = x/(x - 2)$, except that there is a hole at the point $(-2, 1/2)$.

Another common error is to rely on a graphing utility to analyze behavior for very large or very small values of x in the domain. For instance, using the standard viewing window $[-10, 10] \times [-10, 10]$, the graphs of $y = 2^x$ and $y = x^4$ appear to intersect just twice. However, you should recognize that the exponential function $y = 2^x$ ultimately grows faster than the polynomial $y = x^4$. So, the graphs should intersect a third time. You can see this third intersection point if you use the viewing window $[0, 30] \times [0, 90{,}000]$.

One reason limits are so important is that they are used to define the derivative of a function. For instance, you should recognize that

$$\lim_{h\to 0} \frac{\sin(x + h) - \sin x}{h} = \cos x.$$

Try graphing $y_1 = \cos x$ together with $y_2 = (\sin(x + h) - \sin x)/h$ for $h = 0.001$.

Worked Example

Consider the function

$$f(x) = \frac{(x-2)\sin x}{x^3 - 4x}.$$

(a) What is the domain of f?

(b) Find $\lim_{x\to 0} f(x)$.

(c) Find the vertical asymptotes of f.

(d) Calculate $\lim_{x\to -2^-} f(x)$.

(e) Discuss the continuity of f.

SOLUTION

(a) You cannot divide by 0. Setting the denominator $x^3 - 4x = x(x-2)(x+2)$ equal to 0 gives the points where the function is not defined. So, the domain of f is the set of all real numbers except 0, 2, and -2.

(b) Even though f is undefined at $x = 0$, the limit exists as follows.

$$\lim_{x\to 0}\frac{(x-2)\sin x}{x^3-4x} = \lim_{x\to 0}\frac{(x-2)\sin x}{(x-2)(x+2)x} = \left(\lim_{x\to 0}\frac{1}{x+2}\right)\left(\lim_{x\to 0}\frac{\sin x}{x}\right) = \left(\frac{1}{2}\right)(1) = \frac{1}{2}$$

(c) The only vertical asymptote is $x = -2$.

(d) One way to determine the value of a limit is to build a table of values. In this case, you look at values of x close to, but less than -2.

x	-2.1	-2.01	-2.001	-2
$f(x)$	-4.11	-45.03	-454.21	?

From this table you see that $\lim_{x\to -2^-} f(x) = -\infty$.

(e) f is continuous at all points in its domain. f has removable discontinuities at $x = 0$ and $x = 2$, whereas $x = -2$ is a nonremovable discontinuity.

Notes

(b) You can verify this result with a graphing utility. The graph of f appears to pass through the point $(0, 1/2)$. What happens when you try to find the value of f at $x = 0$?

(c) In a similar manner, $\lim_{x\to -2^+} f(x) = \infty$. So, $\lim_{x\to -2} f(x)$ does not exist.

(e) You can see from the graph of f that $x = 0$ and $x = 2$ are removable discontinuities. There are holes in the graph at $(0, 1/2)$ and $[2, (1/8)\sin 2]$. On the other hand, it is impossible to define f at $x = -2$ in order to make the resulting function continuous there.

Name ______________________________

Date ______________________________

Sample Questions

Show all your work on a separate sheet of paper. Indicate clearly the methods you use because you will be graded on the correctness of your methods as well as on the accuracy of your answers.

Multiple Choice

1. Which of the following functions is NOT continuous for all real numbers x?

   (a) $f_1(x) = x^{1/3}$
   (b) $f_2(x) = \dfrac{2}{(x+1)^4}$
   (c) $f_3(x) = |x+1|$
   (d) $f_4(x) = \sqrt{1+e^x}$
   (e) $f_5(x) = \dfrac{x-3}{x^2+9}$

2. Find all the horizontal asymptotes of the function $f(x) = \dfrac{3x}{\sqrt{x^2+x+4}}$.

   (a) $y = 3$ only
   (b) $y = 0$ only
   (c) $y = -3$ only
   (d) $y = 3$ and $y = -3$ only
   (e) $y = 0$ and $y = \sqrt{3}$ only

3. Let g be the following differentiable function.

$$g(x) = \begin{cases} a(x+1)^2, & x \le 0 \\ e^x + b, & x > 0 \end{cases}$$

   Find the values of a and b.

   (a) $a = -\frac{1}{2}, b = -\frac{1}{2}$
   (b) $a = -\frac{1}{2}, b = \frac{1}{2}$
   (c) $a = \frac{1}{2}, b = -\frac{1}{2}$
   (d) $a = \frac{1}{2}, b = \frac{3}{2}$
   (e) $a = -\frac{1}{2}, b = 1$

4. Which of the following functions ultimately grows the fastest as values of x in the domain become very large?

   (a) $f_1(x) = \ln x$
   (b) $f_2(x) = 3^x$
   (c) $f_3(x) = \dfrac{x^3}{x^2+1}$
   (d) $f_4(x) = x^5$
   (e) $f_5(x) = \ln x^5$

Free Response

Consider the function $f(x) = \dfrac{|x|(x-3)}{9-x^2}$.

(a) Evaluate the limit $\lim_{x\to 3} f(x)$.

(b) Determine all vertical asymptotes of f.

(c) Determine all the horizontal asymptotes of f.

(d) Find all the nonremovable discontinuities of f.

SOLUTIONS

Multiple Choice

1. Answer (b). This function is not continuous at $x = -1$, since it is not even defined at that point. The other functions are defined, and continuous, for all values of x.

2. Answer (d). This function has two horizontal asymptotes: $y = 3$ is a horizontal asymptote to the right, and $y = -3$ is a horizontal asymptote to the left. You can find these asymptotes analytically as follows.

$$\lim_{x\to\infty} \frac{3x}{\sqrt{x^2 + x + 4}} = \lim_{x\to\infty} \frac{3}{\sqrt{1 + 1/x + 4/x^2}} = 3$$

$$\lim_{x\to-\infty} \frac{3x.}{\sqrt{x^2 + x + 4}} = \lim_{x\to-\infty} \frac{-3}{\sqrt{1 + 1/x + 4/x^2}} = -3$$

3. Answer (c). The function is differentiable at $x = 0$ and so it must be continuous there. By substituting $x = 0$ into both rules, you obtain $a = b + 1$. Taking the derivative of each rule and equating them at $x = 0$, you also obtain $2a = 1$. So, $a = 1/2$ and $b = -1/2$.

4. Answer (b). For increasing values of x, growth in each of the given functions is as follows. The logarithmic functions f_1 and f_5 grow at the slowest rates. The rational function f_3, which has the line $y = x$ as an asymptote, grows at nearly a constant rate. Functions f_2 and f_4 both grow at the fastest rates; the exponential function f_2 ultimately grows faster than the polynomial function f_4.

Free Response

(a) $\displaystyle \lim_{x\to3} \frac{|x|(x-3)}{9-x^2} = \lim_{x\to3} \frac{|x|(x-3)}{(3-x)(3+x)} = \lim_{x\to3} \frac{-|x|}{3+x} = -\frac{1}{2}$

(b) $x = -3$ is the only vertical asymptote.

(c) To find the horizontal asymptotes you can evaluate the limits at $\pm\infty$.

$$\lim_{x\to\infty} \frac{|x|(x-3)}{9-x^2} = \lim_{x\to\infty} \frac{-|x|}{3+x} = -1$$

$$\lim_{x\to-\infty} \frac{|x|(x-3)}{9-x^2} = \lim_{x\to-\infty} \frac{-|x|}{3+x} = 1$$

So, there are two horizontal asymptotes, $y = 1$ and $y = -1$.

(d) Using the results of parts (a) and (b), you can see that $x = -3$ is the only nonremovable discontinuity. Notice how the graph of f can be used to verify all of the above answers.