10 Writing and Solving Separable Differential Equations and Modeling

Key to Text Coverage

Section	Examples	Exercises	Topics
4.1	1, 7, 8	5–8, 49–64	*Introduction to differential equations*
4.5		35–40	*Differential equations and integration by substitution*
5.2		37–42	*Differential equations and logarithms*
5.4		109–114	*Differential equations and the exponential function*
5.5	7	90–92, 107	*Logistic differential equation*
5.6	1–6	1–78	*Growth and decay models*
5.7	1–10	1–118	*Models using separable differential equations*
7.5		58, 61, 62	*Models using the logistic differential equation*

Formulas

Here are three of the most important differential equations for modeling real life applications, and their solutions.

$y' = ky$	$y = Ce^{kt}$	Growth and Decay Model—page 362
$y' = k(y - y_0)$	$y = y_0 + Ce^{kt}$	Newton's Law of Cooling—page 365
$y' = ky(L - y)$	$y = \dfrac{L}{1 + Ce^{-kLt}}$	Logistic Model—pages 356 and 523

Summary

Some of the most important differential equations that arise in modeling real life phenomena can be solved by the method called **separation of variables.** If y and t are the dependent and independent variables, then the idea is to rewrite the equation so that each variable occurs on only one side of the equation. Then, you can integrate both sides of the equation. For example, for the logistic equation, you would need to use integration by partial fractions.

The constant of proportionality k can generally be positive or negative. For growth and decay models, a positive constant k indicates exponential growth, whereas a negative constant indicates exponential decay, as in radioactivity. The constant C is usually determined by considering some kind of **initial condition.**

A graphing utility is a wonderful tool for verifying a solution to a differential equation, as shown in the example on the next page.

Worked Example

The rate of change of the number of bears $N(t)$ in a population is directly proportional to $1200 - N(t)$, where $t \geq 0$ is the time in years and k is the constant of proportionality. When $t = 0$ the population is 300.

(a) Write down a differential equation that models this growth.

(b) Find $N(t)$ in terms of t and k.

(c) If $N(4) = 600$, find the value of k.

(d) Calculate $N(8)$.

(e) Find $\lim_{t \to \infty} N(t)$.

SOLUTION

(a) $N'(t) = \dfrac{dN}{dt} = k(1200 - N)$

(b) Separate variables and integrate.

$$\frac{dN}{1200 - N} = k\,dt$$

$$\ln|1200 - N| = -kt + C_1$$

$$1200 - N = e^{C_1 - kt} \qquad (0 < N < 1200)$$

$$N = 1200 - Ce^{-kt} \qquad (C = e^{C_1})$$

Since $N(0) = 300$, $C = 900$ and the number of bears is given by

$N(t) = 1200 - 900e^{-kt}$.

(c) $600 = 1200 - 900e^{-4k}$ which implies that $e^{-4k} = 2/3$. Solving for k, you obtain

$$k = -\frac{\ln(2/3)}{4} = \frac{1}{4}\ln\frac{3}{2} \approx 0.101366.$$

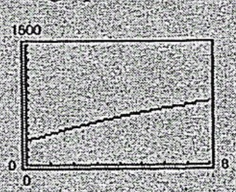
(d) $N(8) = 1200 - 900e^{-8k} = 800$.

(e) Since e^{-kt} tends to zero as t goes to infinity, you have $\lim_{t \to \infty} N(t) = 1200$.

Notes

(a) This model is slightly more complicated than the common growth and decay model, $N'(t) = kN$. Note that the model is similar to that of Newton's Law of Cooling.

(b) It is instructive to graph this function. Enter $N = 1200 - 900e^{-kt}$, where $k = \frac{1}{4}\ln\frac{3}{2} \approx 0.101366$, and use the viewing window $[0, 8] \times [0, 1500]$. Trace along the curve to verify the initial condition $N(0) = 300$ and $N(4) = 600$. Then approximate $N(8)$. What happens to the graph as $t \to \infty$?



Name ____________________

Date ____________________

Sample Questions

Show all your work on a separate sheet of paper. Indicate clearly the methods you use because you will be graded on the correctness of your methods as well as on the accuracy of your answers.

Multiple Choice

1. The half-life of Radium-226 is 1620 years. If 100 grams were present in a sample in the year 1000, how many grams would remain 1000 years later?

   (a) 55.4 (b) 60.3 (c) 65.2 (d) 70.1 (e) 72.3

2. Newton's Law of Cooling states that the rate of change in the temperature y of an object is proportional to the difference between the object's temperature and the temperature y_0 of the surrounding medium. Write the differential equation that models Newton's Law of Cooling.

   (a) $\frac{dy}{dt} = -ky$ (b) $\frac{dy}{dt} = ky$ (c) $\frac{dy}{dt} = y - y_0$

   (d) $\frac{dy}{dt} = k(y - y_0)$ (e) $\frac{dy}{dt} = ky_0y$

3. A certain model for the spread of rumors states that $dy/dt = 3y(3 - 2y)$, where y is the proportion of the population that has heard the rumor at time t. What proportion of the population has heard the rumor when it is spreading the fastest?

   (a) 25% (b) 40% (c) 50% (d) 60% (e) 75%

Free Response

The rate of growth of a certain population $P(t)$ of bacteria is proportional to the product of $P(t)$ and $1000 - P(t)$, where t is the time in hours and k is the constant of proportionality.

(a) Write down a differential equation that models this growth pattern.

(b) Solve the differential equation for P in terms of t and k if $P(0) = 100$.

(c) Find $P(10)$ if $P(5) = 500$.

(d) Describe the long term trend of this population.

SOLUTIONS

Multiple Choice

1. Answer (c). The differential equation that models radioactive decay is

$$y' = \frac{dy}{dt} = ky(k < 0).$$

The solution is $y = Ce^{kt} = 100e^{kt}$. Since the half-life is 1620 years, $50 = 100e^{1620k}$ which implies that

$$\frac{1}{2} = e^{1620k} \text{ or } k = \frac{-\ln 2}{1620} \approx -0.00042787.$$

Finally, $y(1000) = 100e^{1000k} \approx 65.2$ g.

2. Answer (d). The derivative is proportional to the difference of y and y_0.

3. Answer (e). The rate of change of dy/dt is

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}(9y - 6y^2) = 9 - 12y, \quad 0 \le y \le 1.$$

The critical number occurs at $y = 3/4$. Since this derivative is greater than zero to the left of the critical number and less than zero to the right, there is a maximum when $y = 3/4$. So, 75% of the population has heard the rumor when it is spreading the fastest.

Free Response

(a) $P'(t) = \dfrac{dP}{dt} = kP(1000 - P)$ (Logistic differential equation)

(b) Partial fractions are needed to solve the differential equation.

$$\frac{dP}{P(1000 - P)} = k\,dt$$

$$\frac{1}{1000}\left(\frac{1}{P} + \frac{1}{1000 - P}\right)dP = k\,dt \quad \text{(Partial fraction decomposition)}$$

$$\frac{1}{1000}(\ln P - \ln|1000 - P|) = kt + C_1 \quad \text{(Integrate both sides)}$$

$$\ln P - \ln|1000 - P| = 1000kt + C_2 \quad (C_2 = 1000C_1)$$

$$\frac{P}{1000 - P} = Ce^{1000kt} \quad (C = e^{C_2})$$

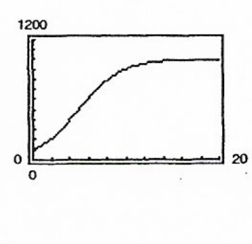
The initial condition $P(0) = 100$ gives $100/(1000 - 100) = 1/9 = C$. Solving for P, you obtain

$$P = \frac{1000e^{1000kt}}{9 + e^{1000kt}} = \frac{1000}{1 + 9e^{-1000kt}}.$$

(c) From the equation

$$\frac{P}{1000 - P} = \frac{1}{9}e^{1000kt}$$

you obtain $1 = (1/9)e^{5000k}$ which gives $k = \ln 9/5000 \approx 0.000439445$.
Finally, $P(10) = 900$.



(d) As t tends to infinity, $e^{-1000kt}$ tends to zero, and the population tends to 1000. Note the distinctive shape of the logistics curve.